DI Physics Hons.

Paper-I

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Relations between the Elastic Constants

The elastic constants are dependent to each other, since any change in the size and shape of a body may be obtained by first changing the size of the body only and then by changing the shape only. Thus, the expressions can be derived, showing the inter-relations between them.

(i) Relations between Y, K and σ:

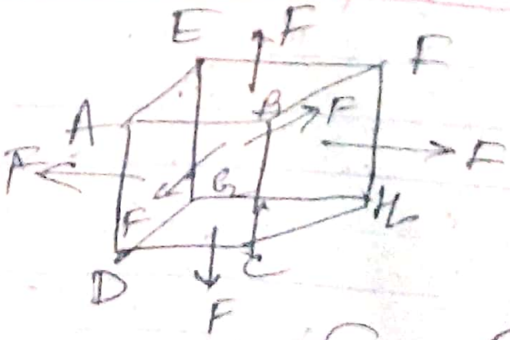

Fig. 1

Let ABCDEFGH represent a cube of unit side. Let us consider a force F, which acts normally and uniformly on each of its six faces in the outward direction.

If α is the increase per unit length per unit tension along the direction of the force, then the elongation produced in each of the edges, namely AB, BF and BC, will be $F.\alpha$. If β is the contraction produced per unit length per unit tension perpendicular to the edges, then contraction produced perpendicular to each of the edge, namely AB, BF and BC, will be $F.\beta$. Thus, the sides of the cube becomes

$$AB = 1+F\alpha - F\beta - F\beta = 1+F(\alpha-2\beta)$$

$$BF = 1+F\alpha - F\beta - F\beta = 1+F(\alpha-2\beta)$$

$$BC = 1+F\alpha - F\beta - F\beta = 1+F(\alpha-2\beta)$$

Hence, the final volume of the cube is

$$AB \times BF \times BC = [1+F(\alpha-2\beta)]^3$$

$$= 1+3F(\alpha-2\beta)$$

[according to Bionomial expansion]

The terms containing higher powers of α and β have been neglected, since α and β are very small quantities.

$\therefore$ change in volume = final volume − initial volume

$$= 1 + 3F(\alpha - 2\beta) - 1 = 3F(\alpha - 2\beta).$$

$$\text{Volumetric strain} = \frac{3F(\alpha-2\beta)}{1} = 3F(\alpha - 2\beta)$$

$$\therefore \text{bulk modulus } K = \frac{\text{normal stress}}{\text{Volumetric strain}} = \frac{F}{3F(\alpha-2\beta)}$$

$$\text{or, } K = \frac{1}{3(\alpha - 2\beta)} \Rightarrow \text{or, } K = \frac{1/\alpha}{3(1-2\beta/\alpha)}.$$

But α is the increase per unit length per unit tension, hence clearly stress = 1 (as the area of any edge of the cube is unity)

$$\text{linear strain} = \frac{\alpha}{1} = \alpha, \quad \therefore Y = 1/\alpha$$

Also,

$$\beta/\alpha = \sigma = \text{Poisson's ratio.}$$

Therefore;

$$K = \frac{Y}{3(1-2\sigma)} \quad \text{---} (i)$$

(ii) Relation between Y, n and σ:

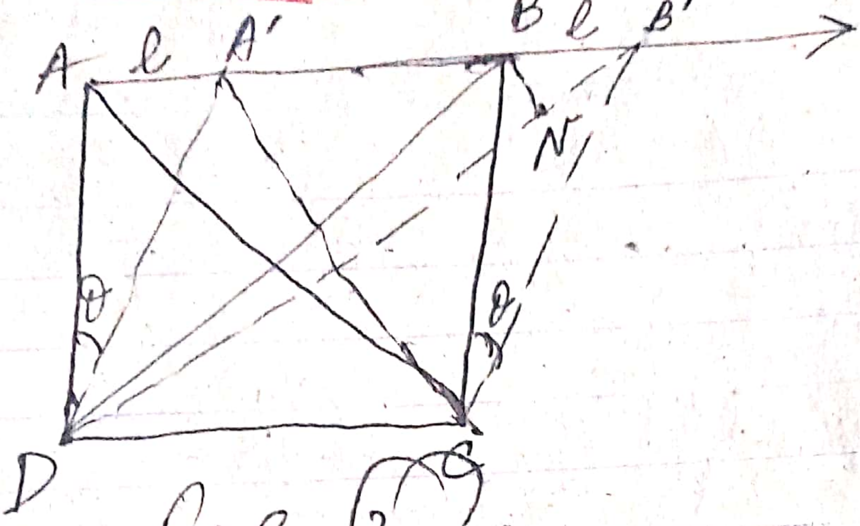

fig. 2.

Let ABCD represent the front face of a cube of side L. A tangential force F is applied on its upper face AB and the bottom face DC is fixed. As a result of this force the cube is sheared to A'B'CD through an angle θ. Then, shearing strain

$$\theta = \frac{AA'}{AD} = \frac{BB'}{BC} = \frac{l}{L},$$

where the displacement

$AA' = BB' = l.$

Shearing stress

$$T = \frac{F}{\text{area of the upper face of the cube}} = \frac{F}{L^2}$$

∴ coefficient of rigidity $n = T/\theta$.

But a shearing stress along AB is equivalent to a tensile stress along DB and an equal compressive stress along AC at right angles to each other.

Let α and β be the longitudinal and lateral strains per unit stress respectively. The extension along diagonal DB due to tensile stress

$$= DB \cdot T \cdot \alpha$$

and extension along diagonal DB due to compression stress along AC

$$= DB \cdot T \, \beta.$$

Total extension along DB

$$= DB \, T (\alpha + \beta) = L\sqrt{2} \cdot T \cdot (\alpha + \beta)$$

Let us draw a perpendicular BN on DB'. Then increase in the length of diagonal DB is practically equal to NB'. As θ is very small, therefore angle AB'C is nearly $90°$ and $\angle BB'N = 45°$.

Thus, $NB' = BB' \cos 45° = \dfrac{BB'}{\sqrt{2}} = \dfrac{l}{\sqrt{2}}$,

$$\therefore L\sqrt{2} \cdot T(\alpha + \beta) = l/\sqrt{2}$$

$$\text{or, } T \cdot \frac{L}{l} = \frac{1}{2(\alpha+\beta)}$$

But $T \cdot \frac{L}{l} = \frac{T}{l/L} = \frac{T}{\theta} = n$,

$$\therefore n = \frac{1}{2(\alpha+\beta)} = \frac{1}{2\alpha(1+\beta/\alpha)}$$

But $\frac{\beta}{\alpha} = \sigma$ and $Y = \frac{\text{Stress}}{\text{longitudinal strain}} = \frac{1}{\alpha}$

[$\because \alpha$ is the longitudinal strain per unit stress.]

Therefore, $n = \frac{Y}{2(1+\sigma)}$ --- (ii)

(iii) Relation between Y, K and n.

From relation (i) $1 - 2\sigma = Y/3K$

From relation (ii) $2+2\sigma = Y/n$

Adding the above two equations, we get

$$3 = \frac{Y}{n} + \frac{Y}{3K} = Y\left(\frac{1}{n} + \frac{1}{3K}\right)$$

$$= Y\left(\frac{3K+n}{3nK}\right)$$

Thus, $$Y = \frac{9nK}{3K+n} \quad \text{---} [iii(a)]$$

This may be written as

$$\frac{9}{Y} = \frac{3K+n}{nK}$$

or, $$\frac{9}{Y} = \frac{3}{n} + \frac{1}{K} \quad \text{---} [iii(b)]$$

(iv) **Relation between K, n and σ:**

From relations (i) and (ii), we have

$$Y = 3K(1-2\sigma)$$

and $$Y = 2n(1+\sigma)$$

Equating the two values of Y, we have

$$3K(1-2\sigma) = 2n(1+\sigma)$$

or, $$3K - 6K\sigma = 2n + 2n\sigma$$

or, $$\sigma(2n+6K) = 3K - 2n$$

Thus, $$\sigma = \frac{3K-2n}{2n+6K} \quad \text{---} (iv)$$